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HEATING OF A HALF-SPACE BY A HEAT SOURCE IN THE SHAPE OF A RECTANGULAR FRAME

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A solution is presented for the inverse nonstationary spatial coefficient problem of heat conduction for a half-space heated through a domain in the shape of a rectangular frame on its surface by an arbitrary heat source with respect to time.

Solutions of nonstationary heat conduction problems obtained [1, 2] for an isotropic half-space under discontinuous boundary conditions of the second kind (a heater in the shape of a circle, ring, square) are used to investigate a set of thermophysical characteristics (TPC) of isotropic materials. To find the TPC set from one experiment there is no necessity to place the sensor within the isotropic body under investigation, i.e., the complex measurements of appropriate thermophysical quantities are realized by nondestructive testing methods [1]. A simple expression is obtained in [3] for the temperature field in the center of an annular heater and the set of TPC of isotropic bodies is investigated on its basis by nondestructive testing methods.

Let us consider an isotropic half-space heated in the domain of the surface $z = 0$ by a heat flux of density $q(\tau)$ in the shape of a rectangular frame $2x_2 \times 2y_2 - 2x_1 \times 2y_1$ (Fig. 1). The rest of the surface is assumed heat insulated. We assume the initial temperature and the temperature at infinity equal to $t_i = \text{const}$, while the first derivatives of the temperature function with respect to the coordinates x, y equal zero. We shall consider those temperature ranges to be considered when the material TPC are temperature independent. In this case, we have the following boundary value problem to determine the excess temperature $\theta = t - t_i$:

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\partial^2\theta}{\partial z^2}=\frac{\dot{\theta}}{a},\quad \lambda\frac{\partial\theta}{\partial z}\bigg|_{z=0}=-q(\tau)\,M(x,\quad y), \tag{1}$$

$$\theta|_{z\to\infty}=0, \tag{2}$$

$$\theta|_{\tau=0}=0,\quad \theta|_{|x|,\,|y|\to\infty}=0,\quad \frac{\partial\theta}{\partial x}\bigg|_{|x|\to\infty}=\frac{\partial\theta}{\partial y}\bigg|_{|y|\to\infty}=0, \tag{3}$$

where $M(x, y) = N_2(x)N_2(y) - N_1(x)N_1(y)$, $\dot{\theta} = \partial\theta/\partial\tau$, $N_i(x) = S(x + x_i) - S(x - x_i)$, $i = 1, 2$.

Applying the Fourier integral transform in x, y and the Laplace in τ to Eq. (1) and boundary conditions (2) under the boundary conditions (3), we obtain, respectively

$$\frac{d^2\bar{\theta}}{dz^2}-\gamma^2\bar{\theta}=0, \tag{4}$$

$$\frac{d\bar{\theta}}{dz}\bigg|_{z=0}=-\frac{\tilde{q}(s)}{\lambda}\hat{M}(\xi,\ \eta),\ \ \bar{\theta}|_{z\to\infty}=0, \tag{5}$$

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where

$$\bar{\theta}=\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\theta(x,\ y,\ z,\ \tau)\exp\left[i(\xi x+\eta y)-s\tau\right]dxdyd\tau,$$

$$\tilde{q}(s)=\int\limits_{0}^{\infty}q(\tau)\exp(-s\tau)\,d\tau,\quad \gamma^2=\xi^2+\eta^2+\frac{s}{a},$$

$$\hat{M}(\xi,\ \eta)=\frac{1}{2\pi}\iint\limits_{-\infty}^{\infty}M(x,\ y)\exp\left[i(\xi x+\eta y)\right]dxdy.$$

The solution of (4) under the boundary conditions (5) has the form

$$\bar{\theta}=\frac{\tilde{q}(s)}{\lambda\gamma}\hat{M}(\xi,\ \eta)\exp(-\gamma z). \tag{6}$$

Going over from the transform to the originals in (6) by using [4, 5], we arrive at the following expression for the excess temperature

$$\theta(x,\ y,\ z,\ \tau)=\frac{1}{4b\sqrt{\pi}}\int\limits_{0}^{\tau}q(\tau-\tau_0)\,\Phi(x,\ y,\ z,\ x_1,\ x_2,\ y_1,\ y_2,\ \tau_0)\,d\tau_0, \tag{7}$$

where

$$\Phi(x,\ y,\ z,\ x_1,\ x_2,\ y_1,\ y_2,\ \tau_0)=\tau_0^{-1/2}\exp\left(-\frac{z^2}{4a\tau_0}\right)[\varphi(x,\ x_2,\ \tau_0)$$
$$\times\,\varphi(y,\ y_2,\ \tau_0)-\varphi(x,\ x_1,\ \tau_0)\,\varphi(y,\ y_1,\ \tau_0)];$$
$$\varphi(x,\ x_i,\ \tau_0)=\operatorname{erf}\frac{x+x_i}{2\sqrt{a\tau_0}}-\operatorname{erf}\frac{x-x_i}{2\sqrt{a\tau_0}};$$

and $b=\lambda/\sqrt{a}$ is the thermal activity.

If the heat flux density at the initial instant changes by a certain quantity q_0 and then remains constant, i.e.,

$$q(\tau)=q_0S_+(\tau), \tag{8}$$

then taking (8) into account we have instead of (7)

$$\theta(x,\ y,\ z,\ \tau)=\frac{q_0S_+(\tau)}{4b\sqrt{\pi}}\int\limits_{0}^{\tau}\Phi(x,\ y,\ z,\ x_1,\ x_2,\ y_1,\ y_2,\ \tau_0)\,d\tau_0. \tag{9}$$

In place of (9) at the center of the heater ($x = y = z = 0$) there will be

$$\theta_0=\theta(0,\ 0,\ 0,\ \tau)=\frac{2q_0}{b\sqrt{\pi}}\left[\sqrt{\tau}\left(\operatorname{erf}\frac{x_2}{2\sqrt{a\tau}}\operatorname{erf}\frac{y_2}{2\sqrt{a\tau}}\right.\right.$$
$$\left.\left.-\operatorname{erf}\frac{x_1}{2\sqrt{a\tau}}\operatorname{erf}\frac{y_1}{2\sqrt{a\tau}}\right)+\frac{1}{\sqrt{\pi a}}\int\limits_{\frac{x_2}{2\sqrt{a\tau}}}^{\infty}\Phi_2(u)\,du-\frac{1}{\sqrt{a\pi}}\int\limits_{\frac{x_1}{2\sqrt{a\tau}}}^{\infty}\Phi_1(u)\,du\right], \tag{10}$$

where $\Phi_i(u) = u^{-1}[x_i\exp(-u^2)\operatorname{erf}(\varepsilon_i u) + y_i\exp(-\varepsilon_i u^2)\operatorname{erf}(u)]$; $\varepsilon_i = y_i/x_i$ ($i = 1, 2$).

Let us rewrite (10) for a frame with identical wall thickness in another form and let us represent it in criterial form

$$\vartheta=\frac{2\theta_0\lambda}{q_0x_2}=\Psi\left(\frac{1}{\sqrt{\mathrm{Fo}}},\ \varepsilon_1\right)-\varepsilon\Psi\left(\frac{\varepsilon}{\sqrt{\mathrm{Fo}}},\ \varepsilon_1\right), \tag{11}$$

where

$$\Psi\left(\frac{1}{\sqrt{\mathrm{Fo}}},\ \varepsilon_1\right)=\frac{2}{\sqrt{\pi}}\int\limits_{\frac{1}{\sqrt{\mathrm{Fo}}}}^{\infty}\operatorname{erf}(u)\operatorname{erf}(\varepsilon_1 u)\frac{du}{u^2};\quad \Psi\left(\frac{\varepsilon}{\sqrt{\mathrm{Fo}}},\ \varepsilon_1\right)=$$

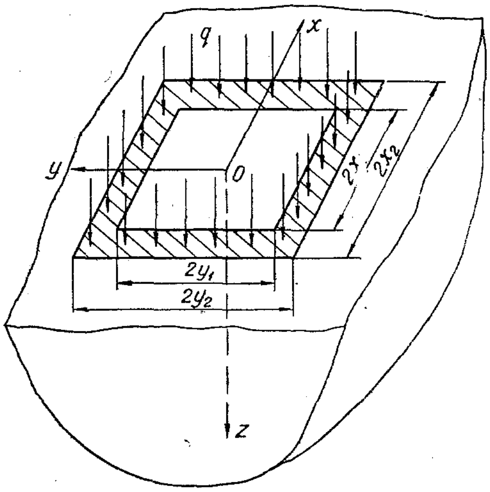

Fig. 1. Model for specimen heating by a heater in the shape of a rectangular frame.

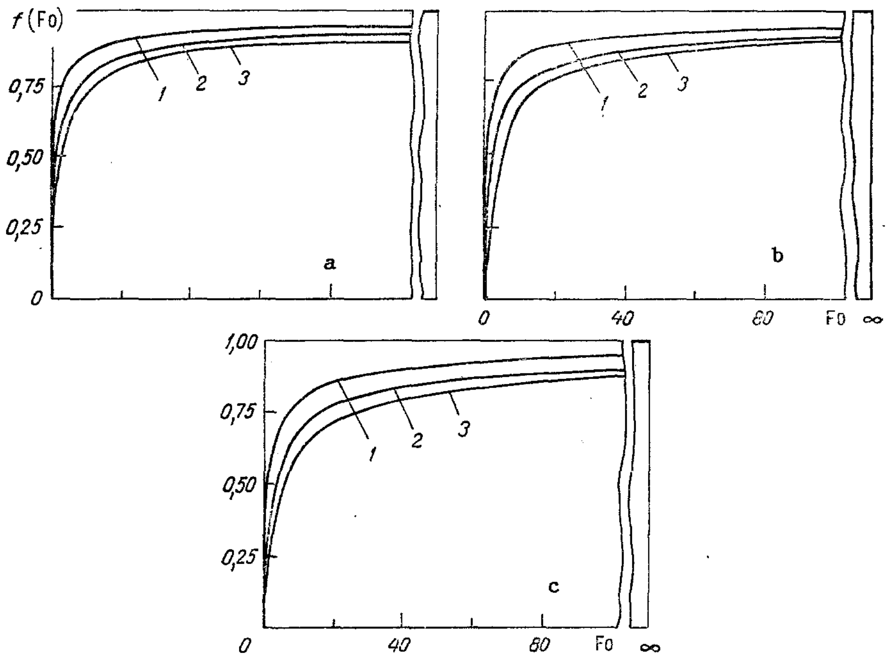

Fig. 2. Change in the function f(Fo) as a function of the Fourier criterion: 1) $\varepsilon = 0$; 2) 0.6; 3) 0.95; a) $\varepsilon_1 = 0.5$; b) 0.7; c) 1.

$$= \frac{2}{\sqrt{\pi}} \int\limits_{\frac{\varepsilon}{\sqrt{\mathrm{Fo}}}}^{\infty} \operatorname{erf}(u) \operatorname{erf}(\varepsilon_1 u) \frac{du}{u^2};$$

$\varepsilon = x_1/x_2 = y_1/y_2$; $\varepsilon_1 = y_1/x_1 = y_2/x_2$; $\mathrm{Fo} = 4a\tau/x_2{}^2$ – is the Fourier criteria.

For $\varepsilon_1 = 1$ the solution for a heater in the form of a square that is obtained by another method in [2], follows from (11).

For a stationary thermal regime ($\tau \to \infty$), we obtain from (10)

$$\vartheta_s = \frac{2\theta_s \lambda}{q_0 x_2} = \frac{4}{\pi} \ln \left[\frac{\left(\dfrac{1 + \sqrt{1 + \varepsilon_2^2}}{\varepsilon_2} \right)^{\varepsilon_2} (\varepsilon_2 + \sqrt{1 + \varepsilon_2^2})}{\left(\dfrac{\varepsilon_1 + \sqrt{1 + \varepsilon_1^2}}{\varepsilon_1^{\varepsilon_1}} \right)^{\varepsilon} (1 + \sqrt{1 + \varepsilon_1^2})^{\varepsilon\varepsilon_1}} \right]. \tag{12}$$

If $\varepsilon_2 = \varepsilon_1$, we have in place of (12)

$$\vartheta_s = \frac{2\theta_s\lambda}{q_0x_2} = 4f_0(\varepsilon, \varepsilon_1). \tag{13}$$

Here

$$f_0(\varepsilon, \varepsilon_1) = \frac{1-\varepsilon}{\pi}\ln\left[\left(\frac{1+\sqrt{1+\varepsilon_1^2}}{\varepsilon_1}\right)^{\varepsilon_1}(\varepsilon_1+\sqrt{1+\varepsilon_1^2})\right].$$

An expression for the heat conduction coefficient

$$\lambda = \frac{2q_0x_2}{\theta_s}f_0(\varepsilon, \varepsilon_1). \tag{14}$$

follows from (13) for moderate values of θ_s for different ε and ε_1.

After substituting (14) into (11) and simple manipulations we find

$$\frac{\theta_0(\mathrm{Fo})}{\theta_s} = f(\mathrm{Fo}) = \frac{1}{4f_0(\varepsilon, \varepsilon_1)}\left[\Psi\left(\frac{1}{\sqrt{\mathrm{Fo}}}, \varepsilon_1\right) - \varepsilon\Psi\left(\frac{\varepsilon}{\sqrt{\mathrm{Fo}}}, \varepsilon_1\right)\right]. \tag{15}$$

The functions f(Fo) are computed by means of (15) for different values of ε and ε_1 that are represented in the form of graphs in Fig. 2.

For fixed values of ε and ε_1 we measure the temperature θ_1 at a definite time τ_1. Knowing the temperatures θ_1 and θ_s we find the value

$$\mathrm{Fo}_1 = \frac{4a\tau_1}{x_2^2}. \tag{16}$$

corresponding to $f(\mathrm{Fo}_1) = \theta_1(\mathrm{Fo}_1)/\theta_s$ from the graph.

Since the τ_1, Fo_1, and x_2 are known, then the thermal diffusivity coefficient is determined in the form

$$a = \frac{\mathrm{Fo}_1 x_2^2}{4\tau_1}. \tag{17}$$

The expression for the heat conduction coefficient has the form (14) for the values taken for ε and ε_1. Therefore, the bulk specific heat coefficient is determined by the known expression $c_v = \lambda/a$.

The method proposed for determining the TPC is characterized by accessibility of the temperature measurements at the center of a heater in the shape of a rectangular frame.

The value of the temperature θ_s for the stationary thermal mode figures in (12). Measurement of this value of the temperature by an IR imager, say, is realized for large values of the Fourier criterion in such a manner that the curve of the temperature change as a function of the time would reach θ_s = const.

NOTATION

λ, a, heat conduction and thermal diffusivity coefficients; x, y, z, space coordinates; τ, time; ξ, η, s, parameters of the Fourier and Laplace integral transforms in x, y, and τ, respectively; erf(ξ), probability integral; $S(\xi)$, $S_+(\xi)$, symmetric and asymmetric unit functions, respectively.

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